Lecture 2: Markov chains

Last Time

0.1. Stochastic process $(X_t)_{t \in I}$

X_t : $\mathcal{X}$ – valued Random Variable.

I : discrete time $I \subseteq \mathbb{N}$

$\mathcal{X}$: discrete state space $\mathcal{X} \subseteq \mathbb{R}$, and $|\mathcal{X}|$ is at most countable.

0.2. Transition probability:

$\mathbb{P}(X_t = x_t \mid (X_s)_{s \le t-1} = (x_s)_{s \le t-1})$

0.3. Markov chain (MC) and Markov property

0.4. Time homogeneity

A Markov chain $(X_t)_{t \in I}$ is time homogeneous if

$\mathbb{P}(X_{n+1} = y \mid X_n = x) = \mathbb{P}(X_1 = y \mid X_0 = x)$, $\forall n \ge 1$, $\forall x, y \in \mathcal{X}$.

0.5. Transition matrix of time homogeneous MC

$P_{xy} = \mathbb{P}(X_t = y \mid X_{t-1} = x)$ $(= \mathbb{P}(X_1 = y \mid X_0 = x))$

Properties: ①. $0 \le P_{xy} \le 1$, $\forall x, y \in \mathcal{X}$.

②. $\sum_{y \in \mathcal{X}} P_{xy} = 1$, $\forall x \in \mathcal{X}$.

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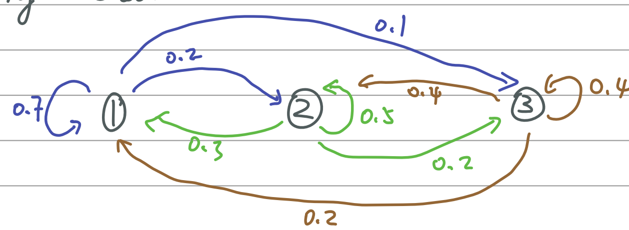
Today:

Part 1°. Visualizing chains / graphs

Ex1. Transition matrix:

$$P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0.7 & 0.2 & \boxed{0.1} \\ 2 & 0.3 & 0.5 & 0.2 \\ 3 & 0.2 & 0.4 & 0.4 \end{array}$$

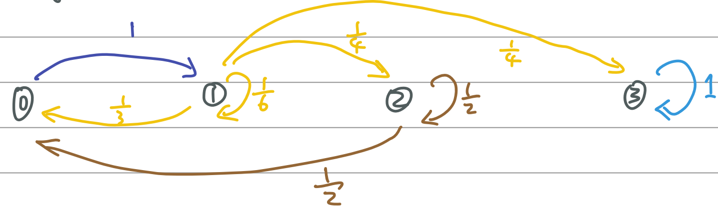
Visualizing chain:



Ex2. Transition matrix:

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 1 & \frac{1}{3} & \frac{1}{6} & \frac{1}{4} & \frac{1}{4} \\ 2 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

Visualizing chain:



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2° Multi-step transition probability

Let $(X_t)_{t\in I}$ be a time homogeneous MC, let P be the transition matrix such that $P_{xy} = \mathbb{P}(X_1 = y \mid X_0 = x)$. Define

$$[P(n, n+m)]_{xy} = \mathbb{P}(X_{n+m} = y \mid X_n = x)$$

be the probability of going from $X_n = x$ to $X_{n+m} = y$.

Q: What is the matrix $P(n, n+m)$?

A: For any $n, m \in \mathbb{N}$, any $x, y \in \mathcal{X}$,

$$
\begin{aligned}
&[P(n, n+m)]_{xy} \\
&= \mathbb{P}(X_{n+m} = y \mid X_n = x) \\
&= \sum_{z\in\mathcal{X}} \mathbb{P}(X_{n+m} = y, X_{n+m-1} = z \mid X_n = x) \\
&= \sum_{z\in\mathcal{X}} \mathbb{P}(X_{n+m} = y \mid X_{n+m-1} = z, X_n = x) \cdot \mathbb{P}(X_{n+m-1} = z \mid X_n = x) \\
&= \sum_{z\in\mathcal{X}} \mathbb{P}(X_{n+m} = y \mid X_{n+m-1} = z) \cdot \mathbb{P}(X_{n+m-1} = z \mid X_n = x) \\
&= \sum_{z\in\mathcal{X}} P_{zy} \cdot [P(n, n+m-1)]_{xz}
\end{aligned}
$$

$$\boxed{\begin{aligned}&\mathbb{P}(A,B \mid C)\\ &= \mathbb{P}(A \mid B,C) \cdot \mathbb{P}(B \mid C)\end{aligned}}$$

Markov Property

$$= [P(n, n+m-1) \cdot P]_{xy}.$$

This implies

$$P(n, n+m) = P(n, n+m-1) \cdot P, \quad \forall n, m \in \mathbb{N}.$$

By mathematical induction or applying the above formula recursively, one has

$$\boxed{P(n, n+m) = P^m, \quad \forall n, m \in \mathbb{N}.}$$

Note:
$[P(n,n+1)]_{xy}$
$= \mathbb{P}(X_{n+1} = y \mid X_n = x)$
$= P_{xy}$,
$\forall n \in \mathbb{N}$, $x, y \in \mathcal{X}$.
$\Rightarrow P(n, n+1) = P$, $\forall n \in \mathbb{N}$.

Thm 1. Let $(X_t)_{t \in I}$ be a time homogeneous Markov chain, then the m-step transition probability

$$\mathbb{P}(X_{n+m} = y \mid X_n = x)$$

is the m-th power of the transition matrix P, evaluated at the row x and column y, i.e., $[P^m]_{xy}$.

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EXAMPLES

Ex 3. Simple Random Walk on $\mathbb{Z}$: $(X_n)_{n\geq 0}$

Let $X_n = \sum_{i=0}^{n} Y_i$, where $Y_0 = 0$, and

$\mathbb{P}(Y_t = 1) = \mathbb{P}(Y_t = -1) = \frac{1}{2}$.

$$\mathbb{P}(X_{n+1} = y \mid X_n = x) = \begin{cases} \frac{1}{2}, & |y-x| = 1 ; \\ 0, & \text{else} . \end{cases}$$

Q: Is this a Markov chain?

Yes.

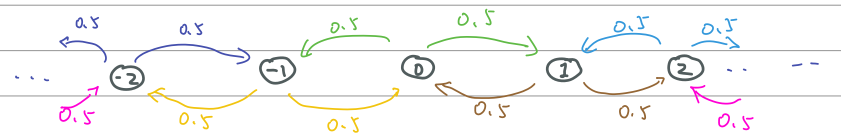
Q: Is this MC time homogeneous?

Yes

Q: What is its transition matrix?

$$P = \begin{array}{c|ccccccc} & \cdots & -2 & -1 & 0 & 1 & 2 & \cdots \\ \hline \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \\ -2 & \cdots & 0 & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ -1 & \cdots & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & \cdots & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots \\ 1 & \cdots & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \cdots \\ 2 & \cdots & 0 & 0 & 0 & \frac{1}{2} & 0 & \cdots \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

Q: What is the visualizing chain?



Ex4 Simple Random Walk on $\mathbb{Z}^2$

$\mathcal{X} = \mathbb{Z}^2$.

$$\mathbb{P}(X_{n+1} = (x_1, y_1) \mid X_n = (x_0, y_0)) = \begin{cases} \frac{1}{4}, & \|(x_1,y_1) - (x_0,y_0)\|_2 = 1; \\ 0, & \text{else}. \end{cases}$$

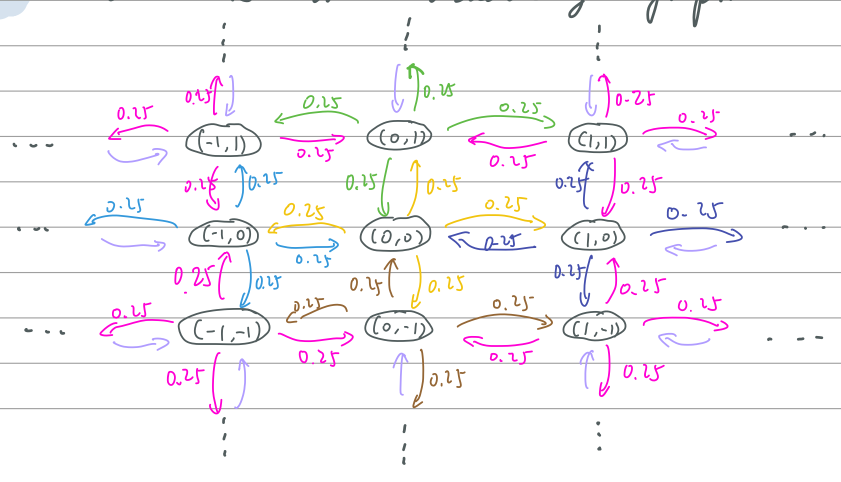
Q: Is this a Markov chain?

Yes.

Q: Is this MC time homogeneous?

Yes

Q: What is the visualizing graph?



Recall

let S, T be two sets, suppose there exists a bijection $f: S \longrightarrow T$, then we say S and T have the same cardinality, denoted as $|S| = |T|$.

e.g. $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$, but $|\mathbb{Q}| = |\mathbb{Z}| = |\mathbb{N}|$.

e.g. $|\mathbb{Z}^2| = |\mathbb{Z}|$.

Ex5. Simple Random Walk on $\mathbb{Z}_4$.

State space $\mathcal{X} = \mathbb{Z}_4 = \{[0], [1], [2], [3]\}$.

$$\mathbb{P}(X_{n+1} = [y] \mid X_n = [x]) = \begin{cases} \frac{1}{2}, & y \equiv x \pm 1 \pmod 4 \\ 0, & \text{otherwise}. \end{cases}$$

Q: Is this a Markov chain?

Yes.

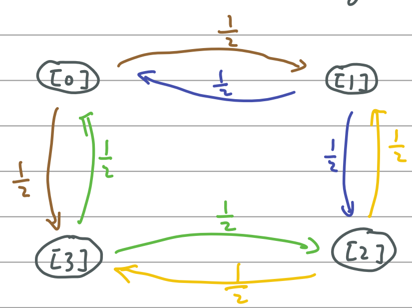
Q: Is this MC time homogeneous?

Yes

Q: What is the transition matrix?

$$P = \begin{array}{c} \\ [0] \\ [1] \\ [2] \\ [3] \end{array} \begin{array}{c} \begin{array}{cccc} [0] & [1] & [2] & [3] \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$

Q: What is its visualizing graph?



This is the end of this Lecture!

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